NASA TECHNICAL NOTE

NASA TN D-2779

c. 1

NASA TN D-2779

ON DEDUCTION OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS FROM THEIR SOLUTIONS

by Murray Tobak

Ames Research Center
Moffett Field, Calif.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1965

NASA TN D-2779

ON DEDUCTION OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS FROM THEIR SOLUTIONS

By Murray Tobak

Ames Research Center
Moffett Field, Calif.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $1.00

ON DEDUCTION OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS FROM THEIR SOLUTIONS

By Murray Tobak
Ames Research Center

SUMMARY

A method is presented for deducing certain nonlinear differential equations from their solutions. The method, relevant to equations of the type $\ddot{x} + \epsilon[\dot{x}g(x^2) + xk(x^2)] + \nu^2 x = 0$, enables the deduction of explicit forms for the nonlinear elements $g(x^2)$ and $k(x^2)$. It should apply whenever the solution can be assumed to be describable by the first approximation of the Kryloff-Bogoliuboff method.

INTRODUCTION

Many fields of mechanics require the study of oscillatory systems governed by nonlinear second-order differential equations. A wide class of such equations yields to treatment by the method of Kryloff and Bogoliuboff (ref. 1). In particular, the first approximation affords a ready means of finding a solution capable of revealing the main features of the oscillatory system.

It does not seem to have been appreciated that the first approximation also lends itself readily to a problem which is the inverse of the usual one; namely, given the solution, find the governing differential equation. This problem arises continually, for example, in experimental studies of aircraft dynamics. Here, the result of an experiment is often a time history of an oscillation. The analyst must extract from this history the form and magnitude of the aerodynamic quantities which are judged to influence the motion. In effect, therefore, the analyst must reconstruct the differential equation which has yielded the observed record. The purpose of this note is to show how the first approximation of the Kryloff-Bogoliuboff method may be adapted to this end for a certain class of equations. For equations of the type $\ddot{x} + \epsilon[\dot{x}g(x^2) + xk(x^2)] + \nu^2 x = 0$, it is shown that, given the solution, explicit forms for the nonlinear elements $g(x^2)$ and $k(x^2)$ are deducible from inversions of Abel integral equations.

A note on precedents: The method proposed here is of such simplicity that it raises the question whether it has not been proposed before. A search for precedents has disclosed one, applicable, however, to another class of equations. In reference 2, G. Plato has shown by a heuristic argument (confirmed by the method proposed here) that the form of the nonlinear element $f(\dot{x})$ of the equation $\ddot{x} + \epsilon f(\dot{x}) + \nu^2 x = 0$ also can be deduced from an Abel integral equation. It is easy to envisage other classes of equations which will yield to the same treatment. It is probable that similar methods, applicable to still other classes of equations, have been or remain to be discovered.

ANALYSIS

The First Approximation

The analysis of reference 1 is concerned with differential equations of the form

$$\ddot{x} + \nu^2 x + \epsilon f(x,\dot{x}) = 0 \tag{1}$$

where ϵ must be a small quantity. Since, with ϵ sufficiently small, x will not deviate greatly from the harmonic solution $x = a \sin(\nu t + \varphi)$, a solution is sought in the form

$$x = a(t)\sin \psi(t) \tag{2}$$

where

$$\psi(t) = \nu t + \varphi(t) \tag{3}$$

It is found that, to a first order in ϵ, $\dot{a}$ and $\dot{\psi}$ must satisfy the relations

$$\dot{a} = -\frac{\epsilon}{2\pi\nu}\int_0^{2\pi} f(a \sin \varphi,\ a\nu \cos \varphi)\cos \varphi \, d\varphi \tag{4}$$

$$\dot{\psi} = \nu + \frac{\epsilon}{2\pi a\nu}\int_0^{2\pi} f(a \sin \varphi,\ a\nu \cos \varphi) \sin \varphi \, d\varphi \tag{5}$$

When equation (1) is given explicitly and the solution for x is desired, equation (2) provides a solution, a(t) and ψ(t) being determinable directly from equations (4) and (5).

The Inverse Problem

Now consider the inverse problem. Let it be assumed that the solution is known in the form of equation (2); that is, $a(t)$ and $\psi(t)$ are known functions. It is desired to find the form of equation (1) under the assumption that the solution is adequately described by the first approximation. This assumption enables one to seek a solution for $f(x,\dot{x})$ through equations (4) and (5).

Consider a class of problems for which it is possible to specify the form of $f(x,\dot{x})$ to the extent shown in equation (6)

$$f(x,\dot{x}) = \dot{x}g(x^2) + xk(x^2) \tag{6}$$

This form is chosen principally on physical grounds, anticipating in particular its pertinence to aerodynamic applications. In reference 3 it is shown that, at least for low reduced frequencies, equation (6) is a general form for the aerodynamic pitching moment when x is identified with angle of attack. The specification that g and k be even functions of x again envisages aerodynamic applications.

Inserting equation (6) in equations (4) and (5) gives

$$\dot{a} = -\frac{\epsilon}{2\pi\nu}\int_0^{2\pi}\cos\varphi[(a\nu\cos\varphi)g(a^2\sin^2\varphi) + (a\sin\varphi)k(a^2\sin^2\varphi)]d\varphi \tag{7}$$

$$\dot{\psi} = \nu + \frac{\epsilon}{2\pi a\nu}\int_0^{2\pi}\sin\varphi[(a\nu\cos\varphi)g(a^2\sin^2\varphi) + (a\sin\varphi)k(a^2\sin^2\varphi)]d\varphi \tag{8}$$

In equation (7), the term involving k, being odd about $\varphi = \pi$, contributes nothing to the integral and may be dropped. In equation (8), on the other hand, the term involving g may be dropped for the same reason. After the change in variable $\xi = a\sin\varphi$, equations (7) and (8) take the form

$$\frac{d}{dt}(a^2) = -\frac{4\epsilon}{\pi}\int_0^a\sqrt{a^2 - \xi^2}\, g(\xi^2)d\xi \tag{9}$$

$$\dot{\psi} = \nu + \frac{2\epsilon}{\pi \nu a^2} \int_0^a \frac{\xi^2 k(\xi^2) d\xi}{\sqrt{a^2 - \xi^2}} \tag{10}$$

With a and ψ being known functions, equations (9) and (10) are a pair of integral equations, the first involving the unknown function $g(\xi^2)$ alone, the second involving the unknown function $k(\xi^2)$.

Inversions

Consider first equation (9). Put $a^2 = \theta$, $\xi^2 = \lambda$, and let

$$\frac{g(\lambda)}{\sqrt{\lambda}} = q(\lambda) \tag{11}$$

so that

$$\dot{\theta} = -\frac{2\epsilon}{\pi} \int_0^\theta \sqrt{\theta - \lambda}\, q(\lambda) d\lambda \tag{12}$$

Since θ is known, it may be assumed that $\dot{\theta}$ is also known, and moreover, that $\dot{\theta}$ can be represented, albeit often only numerically, as a function of θ.[1] Writing

$$\dot{\theta} = F(\theta) \tag{13}$$

and differentiating with respect to θ in equation (12) yields

$$F'(\theta) = -\frac{\epsilon}{\pi} \int_0^\theta \frac{q(\lambda) d\lambda}{\sqrt{\theta - \lambda}} \tag{14}$$

Equation (14) is recognized as Abel's integral equation, which has a particularly simple inversion (see, e.g., ref. 4). It is

[1] In practice, it may be necessary to have a number of records, covering a range of initial conditions, to ensure an adequate representation of this dependency. Changes in initial conditions should only extend the range of the dependency without destroying its uniqueness; the function $\dot{\theta} = F(\theta)$, in other words, should be independent of the initial conditions.

$$q(\theta) = -\frac{1}{\epsilon}\frac{d}{d\theta}\int_0^\theta \frac{F'(\lambda)d\lambda}{\sqrt{\theta - \lambda}} \tag{15}$$

This determines q, and hence g, through equation (11).

Equation (10) is treated similarly. Again putting $a^2 = \theta$, $\xi^2 = \lambda$, and letting

$$\sqrt{\lambda}\, k(\lambda) = p(\lambda) \tag{16}$$

yields

$$\nu\theta(\dot{\psi} - \nu) = \frac{\epsilon}{\pi}\int_0^\theta \frac{p(\lambda)d\lambda}{\sqrt{\theta - \lambda}} \tag{17}$$

With

$$\nu\theta(\dot{\psi} - \nu) = G(\theta) \tag{18}$$

the inversion of equation (17) is

$$p(\theta) = \frac{1}{\epsilon}\frac{d}{d\theta}\int_0^\theta \frac{G(\lambda)d\lambda}{\sqrt{\theta - \lambda}} \tag{19}$$

This determines p, and hence k, through equation (16).

Example

To illustrate the method, we adopt a problem considered in reference 3. Let us assume that from one or several records of the oscillatory system, it has been found possible to represent the solution $x(t)$ as

$$x = \frac{2}{\sqrt{1 - e^{-\epsilon t}(1 - 4/x_o^2)}} \sin\left\{t - \frac{3}{2}\mu\, ln\left[1 + \frac{x_o^2}{4}(e^{\epsilon t} - 1)\right] + \psi_o\right\} \tag{20}$$

Note that x approaches a limit motion as $t \to \infty$. From equation (2) we identify a, ψ, and ν as

$$\left.\begin{aligned} a^2 = \theta &= \frac{4}{1 - e^{-\epsilon t}(1 - 4/x_o^2)} \\ \psi &= t - \frac{3}{2}\mu \, ln\left[1 + \frac{x_o^2}{4}(e^{\epsilon t} - 1)\right] + \psi_o \\ \nu &= 1 \end{aligned}\right\} \tag{21}$$

In the determination of g, it is required first that $\dot{\theta}$ be written as a function of θ. A straightforward calculation yields

$$\dot{\theta} = F(\theta) = -\frac{\epsilon}{4}\theta(\theta - 4) \tag{22}$$

One notes that, as required, $F(\theta)$ is independent of the initial conditions.[2] Inserting $F'(\theta)$ in equation (15) and performing the indicated operations gives

$$q(\theta) = \frac{1}{\sqrt{\theta}}(\theta - 1) \tag{23}$$

whence, from equation (11)

$$g(\theta) = \theta - 1 \tag{24}$$

The determination of k proceeds similarly. Writing $\dot{\psi} - \nu$ as a function of θ yields

$$G(\theta) = -\frac{3}{8}\mu\epsilon\theta^2 \tag{25}$$

Inserting G in equation (19) leads to

$$p(\theta) = -\mu\theta^{3/2} \tag{26}$$

---

[2]The advisability of having a number of records, covering a range of initial conditions, is emphasized in the case of limit motions. Here, for certain initial conditions, θ may be constant over all or a major part of the record. In the present case, for example, when $x_o = 2$, $\theta = 4$. If this were the only record available, one might erroneously conclude that $\dot{\theta}$, and hence g, were identically zero for all θ.

so that, from equation (16)

$$k(\theta) = -\mu\theta \tag{27}$$

Hence, in equation (6)

$$f(x,\dot{x}) = \dot{x}(x^2 - 1) - \mu x^3 \tag{28}$$

so that the equation of motion is, correctly (e.g., ref. 3)

$$\ddot{x} - \epsilon\dot{x}(1 - x^2) + x(1 - \epsilon\mu x^2) = 0 \tag{29}$$

DISCUSSION

The validity of the proposed method for determining the form of the governing differential equation from its solution rests entirely on the assumption that the solution be describable by the first approximation of the Kryloff-Bogoliuboff method. Unfortunately, it is not invariably the case that the appearance of the solution itself is sufficient evidence to justify the assumption a priori; an a posteriori justification is necessary.

Having obtained a form for the differential equation on the basis of the assumption, one should determine that over the range of amplitudes and frequencies of the solution from which the equation has been derived, the inequalities

$$\left.\begin{aligned} \max\,|\epsilon g(x^2)| &\ll \nu \\ \max\,|\epsilon k(x^2)| &\ll \nu^2 \end{aligned}\right\} \tag{30}$$

have been satisfied. Failure to satisfy the inequalities must be taken as an indication that the derived equation may be incorrect. It is advised that in this case, a new evaluation of the nonlinear elements be made on the basis of a smaller range of amplitudes and frequencies of the solution, such that the inequalities are satisfied. If analytic expressions are matched to the nonlinear elements so determined, the resulting differential equation then may be integrated numerically over an extended range of amplitudes and frequencies to check whether the solution it yields matches that part of the solution which had to be excluded in determining the equation.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Feb. 9, 1965

REFERENCES

1. Kryloff, N.; and Bogoliuboff, N.: Introduction to Non-Linear Mechanics. A free translation by Solomon Lefschetz of excerpts from two Russian monographs. First ed., Princeton Univ. Press, Princeton, N. J., or Humphrey Milford, Oxford Univ. Press, London, 1943.

2. Plato, G.: Über das Abklingen von Schwingungen mit schwacher in beliebiger Weise von der Geschwindigkeit abhängiger Dämpfung. Zeitschrift für angewandte Mathematik und Mechanik, Bd. 25/27, Heft 3, Juni, 1947, S. 93-94.

3. Tobak, Murray; and Pearson, Walter E.: A Study of Nonlinear Longitudinal Dynamic Stability. NASA TR R-209, 1964.

4. Titchmarsh, E. C.: Introduction to the Theory of Fourier Integrals. Second ed., Oxford Univ. Press, London, 1948.

NASA-Langley, 1965 A-2032

2/22/85

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546